Anomalous transport in turbulent plasmas and continuous time random walks

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The possibility of a model of anomalous transport problems in a turbulent plasma by a purely stochastic process is investigated. The theory of continuous time random walks (CTRW's) is briefly reviewed. It is shown that a particular class, called the standard long tail CTRW's is of special interest for the description of subdiffusive transport. Its evolution is described by a non-Markovian diffusion equation that is constructed in such a way as to yield exact values for all the moments of the density profile. The concept of a CTRW model is compared to an exact solution of a simple test problem: transport of charged particles in a fluctuating magnetic field in the limit of infinite perpendicular correlation length. Although the well-known behavior of the mean square displacement proportional to $t^{1/2}$ is easily recovered, the exact density profile cannot be modeled by a CTRW. However, the quasilinear approximation of the kinetic equation has the form of a non-Markovian diffusion equation and can thus be generated by a CTRW.

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I. INTRODUCTION

The problem of anomalous transport of particles and of energy in a magnetized plasma is a standing open problem. In particular, in a magnetically confined plasma, such as the one produced in a tokamak or a stellarator, the problem is of outstanding practical importance; it is also of an extraordinary complexity. The concept of anomalous transport was very vague in the beginning (it merely designated whatever mechanism that is different from the classical or neoclassical one). It soon appeared that these anomalous processes were related to unstable fluctuations leading to a turbulent state of the plasma [1]. The most fundamental works in this field use as a starting point a kinetic equation for the distribution function $f^{\alpha}(\mathbf{x},\mathbf{v};t)$ ($\mathbf{x}$ is the position and $\mathbf{v}$ the velocity of a particle of species α) for particles submitted to the action of an electric field $\mathbf{E}(\mathbf{x};t)$ and a magnetic field $\mathbf{B}(\mathbf{x};t)$ and (possibly) to their mutual collisions. The electric and/or the magnetic field are supposed to be fluctuating functions: as a result, the distribution function $f^{\alpha}(\mathbf{x},\mathbf{v};t)$ becomes a random function and the kinetic equation must be treated as a stochastic equation. The determination of the observable quantities then involves a double averaging process: the phase-space average used in statistical mechanics and in kinetic theory in a given realization of the fields, followed by an ensemble average over the realizations of the fluctuating fields.

In spite of (or because of) its attractive features, the theoretical treatment of reasonably realistic kinetic equations derived from first principles is extremely difficult. For this reason, various alternative approaches to transport have been devised. All of these use techniques taken from the mathematical theory of stochastic processes.

We first consider the approach based on the concept of the *Langevin equation* [2–4]. In its simplest version, this is the equation of motion of a test particle moving under the combined influence of the electromagnetic field and the collisions with the other particles of the plasma. The main simplification introduced in the Langevin equation consists in replacing the exact mechanical description of the Coulomb interparticle interactions (which would couple all the particles' equations) by an effective force exerted by the medium (as a whole) on the particle. This force is considered a random quantity, defined by its statistical properties. It is then shown that the solution of the Langevin equation directly leads to a determination of the diffusion coefficient. The Langevin equation can be generalized by considering the electromagnetic field to be also a stochastic quantity: this provides a basis for the anomalous transport [5–9]. Most important, as will be seen below, it opens the door to the treatment of a possible nondiffusive behavior: this will be one of the main goals of the present work.

The Langevin equation approach cannot provide complete information about the transport processes. In particular, it does not determine the shape of the density profile $n(\mathbf{x},t)$ as a function of position and time, information that is very important, especially in the case of nondiffusive processes. The solution of this problem is provided by the Fokker-Planck equation, which can be associated with the Langevin equation (under certain conditions) by a standard procedure [2–4,9–11]. A different version that is even closer to the latter was introduced in our recent work [9] under the name of *hybrid kinetic equation.* The Fokker-Planck or the hybrid kinetic equations can be put to the same uses as a bona fide kinetic equation: their less rigorous dynamical foundation is compensated for by greater simplicity.

All the methods described above can be used whenever the electric and/or the magnetic field are purely stochastic quantities. In a tokamak there may indeed exist regions where the regular toroidal magnetic surfaces are

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completely destroyed, e.g., as a result of tearing instabilities. The previously described methods may be used for the description of transport through such regions. The real magnetic field configuration in a tokamak is, however, incredibly more complex (see, e.g., Ref. [12]): it contains regions of regular tori, magnetic islands of various orders, surrounded by cantori and imbedded in a stochastic sea, in which one may find other structures, such as vortices, holes, etc. An exact description of the transport through such complex media is hopeless. One necessarily has to give up a dynamical picture and appeal even more to statistical concepts.

Although this type of theory is still in its childhood, a class of methods appears to be quite promising. It is based on the concept of random walk, in which one completely gives up the idea of a force driving the motion. The latter is modeled by a succession of random jumps, determined by a set of probabilistic rules. Although the theory of random walks is a very old subject, it has been recently enriched by a series of very important works ([13–17] and references therein), which may open the door to the treatment of highly anomalous contributions of the type described above. The attention of plasma physicists has been directed to these methods by a recent paper [18], which introduced the concept of strange kinetics as a possible new paradigm in the theory of anomalous transport in plasmas.

In the present work we intend to consider the applicability of such methods by studying an academic test problem, which possesses none of the complexities mentioned above, but has the great advantage of being exactly soluble (up to a certain point) and of exhibiting nontrivial anomalous, nondiffusive behavior. A study of this problem was done in Ref. [9] by using the Langevin equations, the hybrid kinetic equation, and the Fokker-Planck equation. Our main purpose here will be to investigate the connection of these "semidynamical" results to a fully stochastic treatment based on a *continuous time random walk* (CTRW) (we use "semidynamical" to indicate that the exact dynamics of the particle collisions is modeled by a stochastic force).

The paper is organized as follows. The problem of anomalous transport due to magnetic fluctuations is defined in Sec. II and the results of Ref. [9] based on the Langevin equations are summarized. The concept of CTRW's and the principal general results are briefly reviewed in Sec. III. In Sec. IV the specific class of CTRW's relevant to our problem are defined under the name of "standard long tail CTRW" (SLT CTRW) and its specific features are described. In particular, the mean square displacement (MSD) and the density profile are known, at least asymptotically, for arbitrary dimensionality. In Sec. V an alternative form of the equation of evolution is derived for the description of the SLT CTRW. It has the very suggestive form of a non-Markovian diffusion equation. Another interesting result derived here is an exact asymptotic expression of all the moments of a SLT CTRW, involving not only their scaling (diffusion exponent), but also the complete numerical coefficient, for arbitrary dimensionality. These general results are then applied to the problem of magnetic turbulence in Sec. VI: it is very easy to construct a SLT CTRW that yields the exact MSD derived from the Langevin equation in Sec. II; it is much less trivial to derive a correct density profile. Two known results of Refs. [5] and [7] are discussed in connection with the SLT CTRW concept. In Sec. VII an exact solution is obtained for the density profile from the hybrid kinetic equation attached to the Langevin equations of the magnetic turbulence problem. This exact density profile, which is a highly nonlinear function of the spectrum of magnetic fluctuations, cannot be described by a CTRW. But the so-called quasilinear approximation of its equation of evolution is precisely equivalent to a SLT CTRW. In addition, some consequences are discussed in Sec. VII. Section VIII presents the conclusions.

II. THE *V*-LANGEVIN EQUATIONS FOR A PARTICLE IN A FLUCTUATING MAGNETIC FIELD

The problem of anomalous transport due to magnetic fluctuations will be the main theme of our present work. It is modeled by considering a charged test particle moving in the presence of a strong magnetic field and undergoing collisions with the surrounding plasma. The latter are modeled as a statistical noise. In the absence of collisions, the motion of the particle is assimilated to the motion of its guiding center, which, to dominant order, follows the magnetic field lines. The description is thus semidynamical in the sense that the exact Coulomb interactions are replaced by a collisional noise.

The magnetic field itself is considered to be of the form

$$\mathbf{B}(z)=B_0[\mathbf{e}_z+b(z)\mathbf{e}_x] \ . \tag{1}$$

B_0 is the strength of the homogeneous reference field, directed along the z axis. $B_0 b(z)$ represents a small fluctuating component perpendicular to the main field. Its complete statistical definition is given below. For simplicity, it is supposed to be directed in the x direction (this is an unessential assumption). More important are the assumptions about the functional dependence: $b(z)$ is assumed to be independent of time ("quenched fluctuations") and also of x,y (the perpendicular correlation length is infinite). As shown in Ref. [9], with these assumptions the problem becomes, in a sense, linear and thus integrable.

We now consider the particle motion. In the present approximation it is tied to the magnetic field lines. In the limit of a very strong field B_0, the perpendicular collisional diffusion is extremely small and can be neglected compared to the parallel collisional diffusion coefficient $\chi_\parallel$. The particle moves in the x direction only in so far as the field lines diffuse in the perpendicular direction. The velocity of the test particle thus obeys the *V*-Langevin (i.e., velocity-Langevin) equations [9]

$$\frac{dx(t)}{dt}=b[z(t)]v(t) \ , \tag{2}$$

$$\frac{dz(t)}{dt}=v(t) \ . \tag{3}$$

Here $v(t)$ is the parallel velocity (denoted by $\eta_\parallel$ in Ref. [9]), which is defined statistically (collisional noise). Equations (2) and (3) form a set of two doubly stochastic equations: they involve the fluctuating magnetic field $b(z)$ and the fluctuating parallel velocity $v(t)$. The fluctuating field $b(z)$ is defined statistically as a spatially homogeneous Gaussian process, whose first two moments are

$$\langle b(z)\rangle_b=0, \quad \langle b(z)b(z+r)\rangle_b=\mathcal{B}(r) . \tag{4}$$

Whenever necessary for clarity, a subscript on an angular bracket denotes the quantity that is averaged over. The function $\mathcal{B}(r)$ is

$$\mathcal{B}(r)=\beta^2 \exp\left[-\frac{r^2}{2\lambda_\parallel^2}\right] . \tag{5}$$

We also consider the Fourier representation of the fluctuating field (the definition of the Fourier transformation is different, though equivalent to the one used in Ref. [9], for an easy comparison with the literature on CTRW)

$$b(z)=\frac{1}{2\pi}\int dk\, e^{-ikz}b(k) . \tag{6}$$

The correlation function of the Fourier components is

$$\langle b(k)b(k')\rangle_b=\mathcal{B}(k)\,\delta(k+k') . \tag{7}$$

The function $\mathcal{B}(k)$ is called the spectral density of the magnetic field fluctuations or, briefly, the spectrum:

$$\mathcal{B}(k)=(2\pi)^{3/2}\lambda_\parallel\beta^2\exp(-\tfrac{1}{2}\lambda_\parallel^2k^2) . \tag{8}$$

In order to represent the effect of the collisions, the fluctuating parallel velocity $v(t)$ is modeled as a Gaussian process with

$$\langle v(t)\rangle_v=0, \quad \langle v(t)v(t+\tau)\rangle_v=\tfrac{1}{2}V_T^2\exp(-\nu|\tau|) , \tag{9}$$

where $V_T=\sqrt{2T/m}$ is the thermal velocity and ν is the collision frequency [9] (T is the plasma temperature and m the mass of the test particle). Finally, it is assumed that the magnetic fluctuations and the velocity fluctuations are statistically independent. This implies the following form for the Eulerian correlation of the field at a fixed position z and the random velocity at an arbitrary time:

$$\langle\langle b(z)v(t)\rangle\rangle_{b,v}=0 , \tag{10}$$

where the double angular brackets denote an average over both b and v.

The stochastic equations (2) and (3), combined with the statistical assumptions (4)–(10) were solved in Ref. [9] with the initial condition $\langle\langle z(0)\rangle\rangle=z_0$, $\langle\langle x(0)\rangle\rangle=x_0$, and $\delta z(0)=\delta x(0)=0$, where $\delta A(t)=A(t)-\langle\langle A(t)\rangle\rangle$. In particular, the MSD $\langle \delta x^2(t)\rangle$ in the x direction can be calculated exactly. From this quantity, a running diffusion coefficient $D(t)$ is derived by the classical formula

$$D(t)=\tfrac{1}{2}\partial_t\langle \delta x^2(t)\rangle . \tag{11}$$

The following expression was obtained in Ref. [9] for the running diffusion coefficient:

$$D(t)=\int_0^t d\tau(2\pi)^{-2}\int dk\, \mathcal{B}(k)\mathcal{Z}(k;\tau) , \tag{12}$$

with

$$\mathcal{Z}(k;\tau)=\left\langle \exp\left[ik\int_t^{t+\tau}d\theta\, v(\theta)\right]v(t+\tau)v(t)\right\rangle_v . \tag{13}$$

The function $\mathcal{Z}(k;\tau)$ was calculated exactly in Ref. [9], as well as the integration over k in Eq. (12):

$$D(t)=2\epsilon\int_0^t d\tau\, F(\nu\tau) , \tag{14}$$

with

$$F(x)=\frac{1}{[1+\gamma\psi(x)]^{1/2}}\left\{e^{-|x|}-\frac{\gamma}{2}\frac{\varphi^2(x)}{1+\gamma\psi(x)}\right\} . \tag{15}$$

Here the following functions have been introduced:

$$\varphi(\xi)=1-e^{-|\xi|}, \quad \psi(\xi)=\xi-\varphi(\xi) . \tag{16}$$

The running diffusion coefficient depends on two parameters: ϵ is a measure of the intensity of the magnetic fluctuations

$$\epsilon=\tfrac{1}{2}\beta^2V_T^2 , \tag{17}$$

whereas γ is interpreted as the square of the ratio of the mean free path $\lambda_{\rm mfp}$ to the parallel correlation length of the magnetic field fluctuations $\lambda_\parallel$; it is therefore a measure of the collisionality of the plasma (a small γ corresponds to a highly collisional plasma)

$$\gamma=\frac{V_T^2}{\nu^2\lambda_\parallel^2}=\frac{\lambda_{\rm mfp}^2}{\lambda_\parallel^2} . \tag{18}$$

The running diffusion coefficient and the MSD were calculated analytically in Ref. [9]:

$$D(t)=\chi_\parallel\beta^2\frac{\varphi(\nu t)}{[1+\gamma\psi(\nu t)]^{1/2}} \tag{19}$$

and

$$\langle \delta x^2(t)\rangle=2\lambda_\parallel^2\beta^2\gamma\{[1+\gamma\psi(\nu t)]^{1/2}-1\} . \tag{20}$$

Here $\chi_\parallel$ is the parallel collisional diffusion coefficient

$$\chi_\parallel=\frac{V_T^2}{2\nu} . \tag{21}$$

It is important to study the behavior of these quantities in two limiting situations. For short time, $\nu t\ll 1$, $\varphi(x)=x+O(x^2)$, and $\psi(x)=x^2/2+O(x^3)$; hence

$$D(t)=\epsilon t, \quad \langle \delta x^2(t)\rangle=\epsilon t^2, \quad \nu t\ll 1 . \tag{22}$$

This is a typically ballistic, nondiffusive behavior. In the opposite limit of long time, $\varphi(x)\sim 1$ and $\psi(x)\sim x$; thus

$$D(t)=\chi_\parallel\frac{\beta^2}{\sqrt{\gamma\nu}}t^{-1/2}, \quad \langle \delta x^2(t)\rangle=4\chi_\parallel\frac{\beta^2}{\sqrt{\gamma\nu}}t^{1/2} . \tag{23}$$

Thus the MSD does not tend asymptotically to a linear function in time. Our model system exhibits strongly anomalous behavior. We define this concept as follows. Assume that for a given statistical system the MSD behaves asymptotically like a power law

$$\langle \delta x^2(t) \rangle = A t^{\alpha} . \tag{24}$$

α is the diffusion exponent (note that there are various definitions for this exponent in the literature). Whenever $\alpha=1$, the system exhibits (normal) diffusive behavior. For every other value of α (or for a dependence different from a power law, such as, e.g., $\ln t$) the system exhibits strongly anomalous behavior.

More specifically, when $0 \le \alpha < 1$, the behavior is called subdiffusive. In this case, in the limit $t \to \infty$, the running diffusion coefficient tends to zero. When $1 < \alpha < \infty$, the behavior is superdiffusive: the asymptotic diffusion coefficient is infinite.

Our present problem provides us with a remarkably simple example of a subdiffusive behavior, with $\alpha=\frac{1}{2}$. For this reason, it will be extensively studied in the following sections.

III. CONTINUOUS TIME RANDOM WALKS

We now go one step further in the modeling of transport processes. In complex problems involving motion of particles through a strongly inhomogeneous and partially disordered medium as described in Sec. I, even the Langevin equation (or the hybrid kinetic equation) becomes prohibitive. In such cases a more radical model could be useful. One gives up the description of the motion by "semideterministic" laws (i.e., a Newton equation with random forces); the evolution is described by a succession of displacements ruled by purely probabilistic prescriptions. This method has been applied to a large variety of problems and has been revived in recent years by important developments [13–18], which are briefly reviewed below.

For full generality, we consider the motion of a test particle in a space of d dimensions. In the CTRW model, the particle performs an instantaneous jump $\mathbf{r}$ of arbitrary length and arbitrary direction at time t and then remains at its new position for a finite time τ, after which it makes a new jump and so on. It is assumed that these jumps are statistically independent. The probability density function (PDF) of a jump described by a vector $\mathbf{r}$ is denoted by $f(\mathbf{r})$. The Fourier transform of this function (called the structure function) will be denoted by $\tilde{f}(\mathbf{k})$. The jumps are performed at random intervals, which must be defined statistically. We thus introduce the waiting time distribution $\psi(t)$, defined as the PDF of a pause of duration t between two successive steps. Alternatively, $\psi(t)$ represents the PDF that a step is taken at a time interval t after the previous one. We also introduce the Laplace transform of this function as well as the corresponding inversion formula

$$\hat{\psi}(s) = \int_0^{\infty} dt\, e^{-st} \psi(t) ,$$
$$\psi(t) = \frac{1}{2\pi i} \int_{\Gamma} ds\, e^{st} \hat{\psi}(s) , \tag{25}$$

where Γ is the usual Bromwich contour in the complex s plane for the inversion of the Laplace transformation, i.e., a parallel to the imaginary axis lying to the right of all singularities of the integrand.

Our main goal will be the calculation of $n(\mathbf{x},t)$: the PDF that a particle starting in $\mathbf{x}=\mathbf{0}$ at time $t=0$ will be in $\mathbf{x}$ at time t. This quantity will be called the *density profile*. In a classical work, Montroll and Weiss [19] derived the following equation for this quantity:

$$n(\mathbf{x},t) = (2\pi i)^{-1} \int_{\Gamma} ds\, e^{st} (2\pi)^{-d} \times \int d^d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{1-\hat{\psi}(s)}{s} \times \frac{1}{1-\hat{\psi}(s)\tilde{f}(\mathbf{k})} . \tag{26}$$

The Montroll-Weiss equation (26) yields the complete solution of the continuous time random walk problem. When the structure function $\tilde{f}(\mathbf{k})$ and the waiting time distribution $\hat{\psi}(s)$ are given, the density profile is determined by a quadrature.

Montroll and Shlesinger [13] showed that the density profile $n(\mathbf{x},t)$ obeys an interesting integro-differential equation

$$\partial_t n(\mathbf{x},t) = \int_0^t d\tau\, \phi(t-\tau) \times \left[-n(\mathbf{x},\tau) + \int d^d\mathbf{x}' f(\mathbf{x}-\mathbf{x}') n(\mathbf{x}',\tau) \right] \tag{27}$$

or, in a Fourier-Laplace representation,

$$s\hat{\tilde{n}}(\mathbf{k},s) - 1 = -\hat{\phi}(s)[1-\tilde{f}(\mathbf{k})]\hat{\tilde{n}}(\mathbf{k},s) , \tag{28}$$

where the kernel $\hat{\phi}(s)$ (in the Laplace representation) is defined as

$$\hat{\phi}(s) = \frac{s\hat{\psi}(s)}{1-\hat{\psi}(s)} . \tag{29}$$

The generalized *master equation* (27) or (28) governs the evolution of the density profile in a CTRW. Its most characteristic feature is its non-Markovian character, both in time and in space: the rate of change of $n(\mathbf{x},t)$ is determined by the past history and by the spatial environment. The effective importance of these features is determined by the range of the functions $\phi(t)$ and $f(\mathbf{x})$.

The following generalization of the central limit theorem for continuous time random walks is rather easily derived [13]: For any CTRW characterized by a transition probability $f(\mathbf{x})$ having at least finite first and second moments $\langle \mathbf{x} \rangle$ and $\langle r^2 \rangle$, respectively [where $r=(\mathbf{x}\cdot\mathbf{x})^{1/2}$] and a waiting time distribution $\psi(t)$ having at least a finite first moment $\langle t \rangle$, the density profile $n(\mathbf{x},t)$ tends to a Gaussian packet for long times and large distances

$$n_G(\mathbf{x},t) = \frac{1}{(2\pi D t)^{d/2}} \exp\left[-\frac{|\mathbf{x}-\mathbf{v}t|^2}{4Dt} \right] , \tag{30}$$

where

$$\mathbf{v} = \frac{\langle \mathbf{x} \rangle}{\langle t \rangle}, \quad D = \frac{1}{2d} \frac{\langle r^2 \rangle}{\langle t \rangle} . \tag{31}$$

The most interesting CTRW's are those corresponding to a definitely non-Gaussian process. In these cases the conditions of the central limit theorem are not satisfied, i.e., either $f(\mathbf{x})$ or $\psi(t)$ or both have a long tail, implying that their first and/or second moments diverge. An important problem of this kind is introduced in the next section.

IV. THE STANDARD LONG-TAIL CONTINUOUS TIME RANDOM WALK

A CTRW is determined by two functions: the transition probability $f(\mathbf{x})$ and the waiting time distribution $\psi(t)$, or equivalently by their Fourier and Laplace transforms $\tilde{f}(\mathbf{k})$ and $\hat{\psi}(s)$, respectively. We now consider a special case of CTRW defined as follows. We take the structure function $\tilde{f}(\mathbf{k})$ to be a symmetrical function, depending only on the absolute value of the wave vector. We assume it to be analytic near $k=0$: this implies the existence of at least the second moment σ^2 of $f(\mathbf{x})$. The form of $\tilde{f}(\mathbf{k})$ near $k=0$ is thus

$$\tilde{f}(\mathbf{k})=1-\frac{1}{2d}\sigma^2k^2+\cdots, \quad k\to 0 . \tag{32}$$

As for the Laplace transform of the waiting time distribution, it is assumed to have the following nonanalytic form near $s=0$:

$$\hat{\psi}(s)=1-\tau_D^\alpha s^\alpha+\cdots, \quad 0<\alpha<1, \quad s\to 0 . \tag{33}$$

We introduced here a characteristic time τ_D, thus $(\tau_D s)$ is a dimensionless quantity. Note that τ_D is *not* to be confused with the average duration of the pauses $\langle t\rangle$, because here $\langle t\rangle\sim(d\hat{\psi}(s)/ds)|_{s=0}\sim s^{\alpha-1}|_{s=0}=\infty$. Such random processes with an infinite average duration between jumps can be imagined [13] as a succession of long pauses followed by bursts of events; very long pauses can occur, but there always is a small but finite probability of an even longer pause. As a result, the average duration between jumps is infinite.

We also consider the inverse Laplace transform of the function $\hat{\psi}(s)$, i.e., the waiting time distribution. The transformation of s^α requires the application of Abelian and Tauberian theorems (see, e.g., [20])

$$\psi(t)=\frac{1}{\tau_D}\frac{\alpha}{\Gamma(1-\alpha)}\left[\frac{t}{\tau_D}\right]^{-1-\alpha}+\cdots, \quad t\to\infty . \tag{34}$$

In view of its importance for our purpose, we call the CTRW defined by Eqs. (32)–(34) the standard long-tail CTRW.

The Montroll-Weiss equation (26) yields the following form of the Fourier-Laplace transform of the density profile in the asymptotic limit $s\to 0$, $k\to 0$:

$$\tilde{\hat{n}}(\mathbf{k},s)=\tau_D^\alpha s^{\alpha-1}\frac{1}{(\tau_D s)^\alpha+\frac{1}{2d}(\sigma k)^2} . \tag{35}$$

From here, the MSD is easily obtained by using a well-known formula

$$\langle r^2(t)\rangle=-\frac{\partial}{\partial\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{k}}\tilde{n}(\mathbf{k},t)|_{k=0} .$$

Performing the appropriate inverse Laplace transformation by using a Tauberian theorem, we obtain

$$\langle r^2(t)\rangle=\frac{1}{\Gamma(1+\alpha)}\sigma^2\left[\frac{t}{\tau_D}\right]^\alpha, \quad 0<\alpha<1 . \tag{36}$$

Thus the behavior of the SLT CTRW is *always subdiffusive*. The MSD is determined by the second moment of the structure function σ^2. We note the important point that the exponent α defining the waiting time distribution (33) turns out to be precisely the *diffusion exponent*. It is also important to note that neither the diffusion exponent α nor the coefficient of t^α depends on the dimensionality d of the space.

A CTRW similar to the SLT CTRW was introduced by Montroll and Shlesinger [13]: they considered, however, the case in which the transition probability $f(\mathbf{x})$ possesses a nonvanishing first moment $\langle\mathbf{x}\rangle$. The result is, in this case, quite different from the previous one: the MSD is determined asymptotically by the first, rather than the second, moment of $f(\mathbf{x})$ and the diffusion exponent is 2α.

The SLT CTRW was studied explicitly by Ball, Havlin, and Weiss [21] and reviewed by Bouchaud and Georges [14]. An important result obtained by Ball, Havlin, and Weiss is an asymptotic analytical expression for the density profile $n(\mathbf{x},t)$ of a SLT CTRW. [Unfortunately some (unimportant) misprints appear in both Refs. [21] and [14]: they are corrected here.] [Note that the density profile depends only on the absolute value of the distance to the origin $r=(\mathbf{x}\cdot\mathbf{x})^{1/2}$: $n(\mathbf{x},t)=n(r,t)$ (as it should, for an isotropic system).] Starting from the Montroll-Weiss equation (26), they performed the inverse Fourier-Laplace transformation, with the result

$$n(r,t)=\sigma^{-d}\left[\frac{\tau_D}{t}\right]^{\alpha d/2}F(q) , \tag{37}$$

where

$$q=\left[\frac{\tau_D}{t}\right]^{\alpha/2}\frac{r}{\sigma} . \tag{38}$$

This is a typical scaling relation. It tells us, in particular, that the density profile depends on r only through the similarity variable q defined in Eq. (38). The function $F(q)$ is defined in terms of a complicated integral, which is not rewritten here. An important consequence of Eq. (37) is the scaling relation for the moments of the density profile. Multiplying Eq. (37) by r^{2p}, integrating over $\mathbf{x}$, and changing the integration variable to $\mathbf{x}\to\sigma(t/\tau_D)^{\alpha/2}\xi$, we easily obtain

$$\langle r^{2p}(t)\rangle=\sigma^{2p}M_p\left[\frac{t}{\tau_D}\right]^{p\alpha} . \tag{39}$$

A quite important feature is the fact that the scaling exponent $p\alpha$ is independent of the dimensionality d of the space. Moreover, the form of the function $F(q)$ does not influence the value of the diffusion exponent α: it only determines the value of the constant M_p. The value of the latter depends on the order p, on the exponent α, and on the dimensionality d. For $p=1$, Eq. (39) reduces, of course, to Eq. (36). In this particular case the constant M_p is independent of d. The present calculation cannot yield the value of the coefficient M_p for arbitrary p: the latter will be obtained in Sec. V.

Although the function $F(q)$ cannot be calculated explicitly in general, its asymptotic behavior has been estimated in Ref. [21] as

$$F(q)\sim\exp(-\tfrac{1}{4}q^{\delta}),\quad \delta=\frac{2}{2-\alpha},\quad q\to\infty \ . \tag{40}$$

This characteristic "stretched exponential" behavior reduces, as it should, to the Gaussian profile when $\alpha=1$ (i.e., in the diffusive case).

We now consider the one-dimensional SLT CTRW, i.e., the case $d=1$. It turns out that this case is particularly interesting for our applications; on the other hand, it is possible to derive completely explicit formulas for this case. Ball, Havlin, and Weiss [21] obtained the expression

$$n(x,t)=\frac{1}{\sigma}\frac{1}{2^{3-\alpha}\Gamma\left[\dfrac{4-\alpha}{2}\right]}\left[\frac{t}{\tau_D}\right]^{-\alpha/2} \times\exp\left[-\frac{q^{2/(2-\alpha)}}{4}\right],\quad q\gg1,\ d=1 \ . \tag{41}$$

This form of the one-dimensional density profile of the SLT CTRW will be used in forthcoming sections.

V. THE NON-MARKOVIAN DIFFUSION EQUATION

We now consider the problem of the equation of evolution satisfied by the SLT CTRW. In Sec. III the Montroll-Shlesinger master equation (27) and (28) for a general CTRW [13] was recalled. It appeared as a linear, non-Markovian equation for the density profile $n(\mathbf{x},t)$. Our purpose here is to derive its specific form for the standard long-tail CTRW and study its relation to the ordinary diffusion equation.

We consider thus again the SLT CTRW defined by Eqs. (32) and (33). Its asymptotic Fourier-Laplace density profile was obtained, in the limit $s\to0$, $k\to0$, in Eq. (35). It is easily checked that this function satisfies the master equation (28) with the kernel $\hat{\phi}(s)$ defined by Eq. (29):

$$\hat{\phi}(s)=\frac{1}{\tau_D}\frac{1-(\tau_D s)^{\alpha}}{(\tau_D s)^{\alpha-1}} \ ,$$

which is simplified by retaining only the leading term as $s\to0$:

$$\hat{\phi}(s)=\frac{1}{\tau_D}(\tau_D s)^{1-\alpha} \ . \tag{42}$$

The Fourier-Laplace master equation (28) reduces to

$$s\tilde{\hat{n}}(\mathbf{k},s)-1=-\hat{\phi}(s)\frac{1}{2d}(\sigma k)^2\,\tilde{\hat{n}}(\mathbf{k},s) \ . \tag{43}$$

The inverse Fourier transform of this equation is

$$s\hat{n}(\mathbf{x},s)-\delta(\mathbf{x})=\frac{\sigma^2}{2d}\hat{\phi}(s)\nabla^2\hat{n}(\mathbf{x},s) \ . \tag{44}$$

This looks very much like a Laplace-transformed diffusion equation [we recall that the initial value of the density profile is $n(\mathbf{x},t=0)=\delta(\mathbf{x})$]. However, the diffusion coefficient in Laplace space is not a constant but a function of s. As a result, the inverse Laplace transform of Eq. (44) is an integral equation in time

$$\partial_t n(\mathbf{x},t)=\frac{\sigma^2}{2d}\int_0^t d\tau\,\phi(\tau)\nabla^2 n(\mathbf{x},t-\tau) \ . \tag{45}$$

Our main conclusion at this point is that the density profile of a standard long-tail CTRW obeys a linear, non-Markovian integro-differential equation. Given the obvious resemblance of this equation to the diffusion equation, we call it the *non-Markovian diffusion equation.* Thus the rate of change of the density profile at time t is influenced by its past history. Moreover, given the long tail of the kernel $\phi(t)$, the domain of effective influence extends far into the past. It is this feature that is responsible of the non-Gaussian character of the density profile and of the anomalous transport law.

In order to calculate the kernel $\phi(t)$ we must proceed with some care in order to avoid typical difficulties related to the long tail. We note that $\hat{\phi}(s)$ can be simply related to the Laplace transform of the waiting time distribution $\hat{\psi}(s)$, defined in Eq. (33):

$$\hat{\varphi}(s)=\frac{1}{\tau_D}(\tau_D s)^{1-\alpha}=\frac{1}{\tau_D}[1-\hat{\psi}_{1-\alpha}(s)] \ , \tag{46}$$

where $\hat{\psi}_{1-\alpha}(s)$ denotes the function $\hat{\psi}(s)$ in which the index α is changed to $1-\alpha$. We now evaluate the inverse Laplace transform as explained in Sec. IV. The transform of 1 is $\delta(t)$, which can also be written, for convenience, as $\tau_D\delta(t/\tau_D)$. The transform of the second term is provided by Eq. (34):

$$\phi(t)=\frac{1}{\tau_D^2}\left[\delta\left[\frac{t}{\tau_D}\right]-\frac{1-\alpha}{\Gamma(\alpha)}\left[\frac{\tau_D}{t}\right]^{2-\alpha}\right] . \tag{47}$$

The δ function is zero for $t>0$; we shall, however, keep it here because it will help us in the forthcoming derivation. Substituting Eq. (47) into (45) we obtain the following form for the equation of evolution:

$$\partial_t n(\mathbf{x},t)=D_0\nabla^2 n(\mathbf{x},t) -H_0\int_0^t d\tau\left[\frac{\tau_D}{\tau}\right]^{2-\alpha}\nabla^2 n(\mathbf{x},t-\tau) \ , \tag{48}$$

where

$$D_0=\frac{\sigma^2}{2d\tau_D},\quad H_0=\frac{1-\alpha}{\Gamma(\alpha)}\frac{1}{\tau_D}D_0 \ . \tag{49}$$

Equation (48) describes the evolution as a superposition of an "ordinary" diffusion term, with a diffusion coefficient D_0, and a non-Markovian process represented by the second term. The characteristic feature of the kernel in the latter is its *long tail:* it decreases very slowly, as an inverse power law (note that $2-\alpha>0$). A very interesting feature appears in the definition of the coefficient H_0: being proportional to $(1-\alpha)$, this coefficient vanishes when $\alpha=1$. Thus, in the diffusive case $\alpha=1$, the non-Markovian term in Eq. (48) disappears and we are left with a pure diffusion equation.

As it stands, however, Eq. (48) does not really make sense; indeed, the kernel $\tau^{-2+\alpha}$ diverges at the lower limit $\tau=0$. This is not surprising because the expression of $\psi(t)$ [Eq. (34)] and hence the second term of $\phi(t)$ [Eq. (47)], are *asymptotic expressions,* valid for long times: they are certainly inapplicable for $\tau\to 0$. In order to cure this difficulty, we introduce a lower cutoff $\tau_{\min}$ in the integral of Eq. (48). This parameter should be determined in such a way as to satisfy a well-defined criterion: its choice will be the object of the forthcoming discussion. The equation will thus be written in the final form

$$\partial_t n(\mathbf{x},t)=D_0\nabla^2 n(\mathbf{x},t) - H_0\int_{\tau_{\min}}^{t} d\tau \left[\frac{\tau_D}{\tau}\right]^{2-\alpha} \nabla^2 n(\mathbf{x},t-\tau) . \quad (50)$$

In order to define the criterion mentioned above, we consider the moments of this equation, beginning with the second moment, i.e., the MSD. This moment was determined independently in Eq. (36) and was shown to scale asymptotically as $\langle r^2(t)\rangle \sim t^\alpha$; thus $\partial_t\langle r^2(t)\rangle \sim t^{\alpha-1}$. We now calculate the rate of change of the MSD by using Eq. (50). Multiplying both sides by r^2, integrating by parts over $\mathbf{x}$, and using Eq. (49), we obtain

$$\partial_t\langle r^2(t)\rangle = 2dD_0\left\{1+\frac{1}{\Gamma(\alpha)}\frac{t^{\alpha-1}-\tau_{\min}^{\alpha-1}}{\tau_D^{\alpha-1}}\right\} . \quad (51)$$

This expression contains a term that has the correct scaling $\sim t^{\alpha-1}$ and a term that is constant, which would dominate for $\alpha<1$: the latter term is thus spurious. But, we can use our freedom for defining the cutoff $\tau_{\min}$ in such a way as to annul this term: this leads to the value

$$\tau_{\min}=[\Gamma(\alpha)]^{-1/(1-\alpha)}\tau_D . \quad (52)$$

With this choice of the cutoff, we obtain from Eqs. (51) and (49)

$$\langle r^2(t)\rangle = 2dD_0\tau_D\frac{1}{\alpha\Gamma(\alpha)}\left[\frac{t}{\tau_D}\right]^\alpha=\frac{\sigma^2}{\Gamma(1+\alpha)}\left[\frac{t}{\tau_D}\right]^\alpha .$$

This result is precisely identical to Eq. (36) obtained by an independent method.

This result shows that the diffusion term in Eq. (50) is, in a sense, "spurious" it compensates for the effect originating from the inaccurate short-time domain: the really physical mechanism of evolution is in the long-tail non-Markovian operator. However, in the diffusive case $\alpha=1$, the non-Markovian term vanishes and we are left with a pure diffusion equation, as expected.

The validity of Eq. (50) is, however, not yet satisfactorily settled. Indeed, although it yields the correct second moment of the density profile, we should also wonder about the higher-order moments.

In order to determine the higher-order moments, we first note the identity (valid for arbitrary dimensionality)

$$\nabla^2 r^{2p}=2p(2p+d-2)r^{2p-2} . \quad (53)$$

We then derive from Eq. (50) [by the same procedure as used for Eq. (51)] the following recursion relation for the moments:

$$\partial_t\langle r^{2p}(t)\rangle = 2p(2p+d-2)D_0\left\{\langle r^{2p-2}(t)\rangle - \frac{1-\alpha}{\Gamma(\alpha)}\frac{1}{\tau_D}\int_{\tau_{\min}}^{t} d\tau\left[\frac{\tau_D}{\tau}\right]^{2-\alpha}\langle r^{2p-2}(t-\tau)\rangle\right\} . \quad (54)$$

The scaling of the moments was derived in Eq. (39): $\langle r^{2p}(t)\rangle = C_p t^{p\alpha}$. Substituting this form into Eq. (54), we find

$$p\alpha C_p t^{p\alpha-1}=2p(2p+d-2)D_0\left\{C_{p-1}t^{(p-1)\alpha}-\frac{1-\alpha}{\Gamma(\alpha)}\tau_D^{1-\alpha}C_{p-1}\int_{\tau_{\min}}^{t} d\tau\,\tau^{\alpha-2}(t-\tau)^{(p-1)\alpha}\right\} . \quad (55)$$

The value of the integral can be obtained, e.g., by the MATHEMATICA computer program; the result involves a hypergeometric function whose value, for $(\tau_{\min}/t)\ll 1$, is 1. Equation (56) then reduces to

$$p\alpha C_p t^{p\alpha-1}=2p(2p+d-2)D_0C_{p-1}\left\{t^{(p-1)\alpha}-(1-\alpha)\tau_D^{\alpha-1}\frac{\Gamma(\alpha-1)\Gamma[1+\alpha(p-1)]}{\Gamma(\alpha)\Gamma(p\alpha)}t^{p\alpha-1}-\left[\frac{\tau_{\min}}{\tau_D}\right]^{\alpha-1}\frac{1}{\Gamma(\alpha)}t^{(p-1)\alpha}\right\} . \quad (56)$$

We see again on the right-hand side two terms proportional to $t^{(p-1)\alpha}$ that do not have the same scaling as the left-hand side. Their coefficient must therefore be set equal to zero; thus

$$1-\left[\frac{\tau_{\min}}{\tau_D}\right]^{\alpha-1}\frac{1}{\Gamma(\alpha)}=0 .$$

This yields the *same value* for $\tau_{\min}$ as Eq. (52). This is a very important result: it guarantees that the non-Markovian diffusion equation (50), combined with the definition (52) for the cutoff, yields the correct values for *all the moments* of the density profile and hence for the profile itself.

It is now easily seen that, using some identities for the Γ function as well as the definition (49), Eq. (56) reduces to

$$C_p = \frac{\sigma^2}{d}(p-1)(2p+d-1)\frac{\Gamma[\alpha(p-1)]}{\Gamma(\alpha p)}\tau_D^{-\alpha}C_{p-1} . \quad (57)$$

This recurrence relation is easily solved. One can check by direct substitution that the moment of order $2p$ has the form (39), with the following form for the coefficient:

$$M_p = \frac{d(d+2)\cdots(d+2p-2)}{d^p}\frac{(p-1)!}{\alpha\Gamma(p\alpha)} . \quad (58)$$

This coefficient gives the final solution for all the moments of the SLT CTRW, for arbitrary diffusion exponent α $(0<\alpha\leq 1)$, arbitrary order p, and arbitrary dimensionality d. It is hardly necessary to emphasize the fact that all the moments are independent of the cutoff $\tau_{\min}$.

It is interesting to derive some special cases. Consider first the case of the MSD, i.e., the case $p=1$, in arbitrary dimensionality. Recalling that $\alpha\Gamma(\alpha)=\Gamma(1+\alpha)$, we see that Eq. (58) then reduces to

$$M_1 = \frac{1}{\Gamma(1+\alpha)}, \quad p=1 . \quad (59)$$

This result agrees with Eq. (36). Thus the MSD is the only moment whose value is independent of the dimensionality d.

Next, we consider the *diffusive* case $\alpha=1$ for arbitrary dimensionality. In this case we have $\alpha\Gamma(p\alpha)=\Gamma(p) =(p-1)!$ and Eq. (58) reduces to

$$M_p = \frac{d(d+2)\cdots(d+2p-2)}{d^p}, \quad \alpha=1 . \quad (60)$$

This is a quite interesting case. In order to derive the moments in the diffusive case, one may start from Eq. (50) in which the non-Markovian term is deleted because $H_0 \sim (1-\alpha)$ [Eq. (49)]. We are then left with an ordinary diffusion equation, from which we derive a recursion equation, which is simply Eq. (54) without the non-Markovian term:

$$\partial_t\langle r^{2p}(t)\rangle = 2p(2p+d-2)D_0\langle r^{2p-2}(t)\rangle . \quad (61)$$

For $p=1$, this equation reduces to $\partial_t\langle r^2(t)\rangle = 2dD_0$, with the well-known diffusive solution for the MSD: $\langle r^2(t)\rangle = 2dD_0 t$. From this starting point, the recursion relation (61) is solved by successive iteration, yielding a solution identical to Eq. (58), with $\alpha=1$. The surprising feature is that in the derivation of Eq. (58), at the step (56), the contribution of the diffusive term, i.e., the first term on the right-hand side of Eq. (56), is exactly *canceled* by the third term. Hence the expression (58), valid for arbitrary $\alpha<1$, results entirely from the non-Markovian term in the equation of evolution. Nevertheless, when extrapolated to $\alpha=1$, the result connects smoothly to the result obtained from the "pure" diffusion equation.

Another special case that will be of special interest for us in the next section is the one-dimensional case $(d=1)$ for arbitrary α. In this case we note that

$$\begin{aligned}\frac{d(d+2)\cdots(d+2p-2)(p-1)!}{d^p} &= 1\times3\times5\cdots\times(2p-1)\times1\times2\times3\cdots(p-1)\\ &= 1\times3\cdots\times(2p-1)\times\frac{2\times4\times\cdots\times(2p-2)}{2^{p-1}}\\ &= \frac{(2p-1)!}{2^{p-1}} ;\end{aligned}$$

thus

$$M_p = \frac{(2p-1)!}{2^{p-1}}\frac{1}{\alpha\Gamma(\alpha)}, \quad d=1 . \quad (62)$$

VI. MAGNETIC SUBDIFFUSION AND CTRW

We now return to the problem of magnetic turbulence treated in Sec. II. This problem was treated semidynamically by starting from the V-Langevin equations describing the motion of guiding centers in the presence of a fluctuating magnetic field and of collisional diffusion in a parallel direction. In the limit of an infinite perpendicular correlation length of the stochastic field, it was shown that the motion of the particles is subdiffusive and the MSD could be calculated exactly. Quite naturally, we may ask ourselves whether this subdiffusive behavior could be described by a continuous time random walk. Given that many features of the present problem can be calculated analytically, it appears as a remarkable "laboratory" for testing the CTRW model.

The starting point of the analysis is Eq. (22) for the MSD: it shows that this quantity is asymptotically proportional to $\sqrt{t}$. In view of our study of CTRW's, it appears natural to try to interpret this process in terms of a standard long-tail CTRW, which always describes a subdiffusive process. In order to define a SLT CTRW we need to specify four parameters [see Eqs. (32) and (33)]: α, d, τ_D, and σ. These must be related to the parameters at our disposal in our problem, i.e., the collisional parallel diffusion coefficient $\chi_\parallel$, and the magnetic fluctuation characteristics, i.e., the parallel correlation length $\lambda_\parallel$ and the intensity of the fluctuations β.

(i) The value of the diffusion exponent is obvious: $\alpha=\frac{1}{2}$.

(ii) The value of the dimensionality requires a short discussion. From the point of view of fusion plasma physics, the main interest in transport theory lies in the study of the transport across the toroidal magnetic surfaces of a confined plasma, i.e., in the radial direction ρ of a toroidal coordinate system. In our shearless slab geometry, which is a local approximation to this situation, the radial coordinate is simulated by the x direction, perpendicular to the main field. Thus our interest will be in the *one-dimensional transport* processes in the x direction. Nevertheless, the two-dimensional process will also be briefly considered below.

(iii) In order to define a characteristic time we could first think of the inverse collision frequency ν^{-1}. This, however, does not properly describe the behavior of the particles that remain tied to the magnetic field (because the perpendicular collisional diffusion is neglected), the latter being coherent over a distance of order $\lambda_\parallel$. The natural definition of a typical time scale is the time it takes for a localized bunch of particles to diffuse collisionally over a distance $\lambda_\parallel$ in the z direction:

$$\tau_D=\frac{\lambda_\parallel^2}{2\chi_\parallel}=\frac{\lambda_\parallel^2\nu}{V_T^2}=\frac{1}{\gamma\nu} . \tag{63}$$

The second form is obtained by using Eq. (23) and the third form by Eq. (18). Thus the characteristic time τ_D differs from ν^{-1} by the factor γ^{-1}, which is large when the plasma is highly collisional.

(iv) Having defined the characteristic time τ_D, the remaining parameter σ is obtained by comparing the expression (23) of the MSD obtained from the V-Langevin equation with the expression (36) obtained from the CTRW (note that for $\alpha=\frac{1}{2}$, $[1/\alpha\Gamma(\alpha)]=2/\sqrt{\pi}$):

$$\langle \delta x^2(t)\rangle=4\chi_\parallel\frac{\beta^2}{(\gamma\nu)^{1/2}}t^{1/2}=\frac{2}{\sqrt{\pi}}\sigma^2\left[\frac{t}{\tau_D}\right]^{1/2} . \tag{64}$$

Using Eq. (11) we obtain the corresponding running diffusion coefficient

$$D(t)=\frac{1}{2\sqrt{\pi}}\frac{\sigma^2}{\tau_D}\left[\frac{t}{\tau_D}\right]^{-1/2} . \tag{65}$$

Using, on the one hand, Eqs. (18) and (21) and, on the other hand, Eq. (63), we find

$$\sigma^2=\sqrt{\pi}\beta^2\lambda_\parallel^2 . \tag{66}$$

Thus the characteristic perpendicular jump length σ of the CTRW is determined by the characteristics of the magnetic fluctuations, in particular by their intensity β and their correlation length $\lambda_\parallel$ [Eq. (5)]. This is clear because the only mechanism producing a radial displacement of the particles (tied to the magnetic lines) is the radial diffusion of these lines themselves.

We collect these results in the following list defining the SLT CTRW modeling the magnetic subdiffusive motion of the plasma:

$$\alpha=\tfrac{1}{2},\quad d=1,\quad \tau_D=\frac{\lambda_\parallel^2}{2\chi_\parallel}=\frac{1}{\gamma\nu},\quad \sigma^2=\sqrt{\pi}\beta^2\lambda_\parallel^2 . \tag{67}$$

It is thus very easy to adapt a SLT CTRW to our test problem and to note that its characteristic parameters have a reasonable physical meaning. The MSD is, however, a rather weak criterion, as many different models may lead to the same MSD. A much more sensitive test is the *density profile,* which will presently be considered.

The SLT CTRW model yields a density profile that was derived under very general conditions by Ball, Havlin, and Weiss [21] [Eq. (41)]. For $\alpha=\frac{1}{2}$ and $d=1$, this quantity is

$$n_{\rm SLT}(x,t)=\frac{1}{\sigma}\frac{1}{2^{5/2}\Gamma(\frac{7}{4})}\left[\frac{\tau_D}{t}\right]^{1/4}\exp\left[-\frac{q^{4/3}}{4}\right] ,\qquad \alpha=\tfrac{1}{2},\ d=1 . \tag{68}$$

The similarity variable q, defined by Eq. (38), reduces here to

$$q=\left[\frac{\tau_D}{t}\right]^{1/4}\frac{r}{\sigma} . \tag{69}$$

Alternatively, the SLT profile may be characterized as a solution of the non-Markovian diffusion equation, which reduces in the present case to

$$\partial_t n_{\rm SLT}(x,t)=D_0\nabla_x^2 n_{\rm SLT}(x,t)-H_0\int_{\pi^{-1}}^{t}d\tau\left[\frac{\tau_D}{\tau}\right]^{3/2}\nabla_x^2 n_{\rm SLT}(x,t-\tau) , \tag{70}$$

with

$$D_0=\frac{\sigma^2}{2\tau_D},\quad H_0=\frac{1}{2\sqrt{\pi}\tau_D}D_0 . \tag{71}$$

These results will now be compared with the semidynamical model. The latter can be treated in several "levels of sophistication," which will be successively considered below.

A first result is obtained by translating into an equation of evolution the argument described qualitatively by Rechester and Rosenbluth [5]. The magnetic fluctuations produce a spatial diffusion of the magnetic lines [5,22,23] that can be described by a diffusion equation in which the role of the time is played by the coordinate z:

$$\frac{\partial}{\partial z}\bar{n}(x,z)=d_m\nabla^2\bar{n}(x,z) . \tag{72}$$

This is a purely geometrical problem. $\bar{n}(x,z)$ is the PDF for a point starting at "time" $z=0$ on a certain magnetic line, to be found at a perpendicular deviation x at "time" z. The magnetic diffusion coefficient d_m can be calculated exactly in the limit $\lambda_\perp\to\infty$ by solving the corresponding Langevin equation: it yields a well-known result [5,22,23]

$$d_m=\left[\frac{\pi}{2}\right]^{1/2}\beta^2\lambda_\parallel . \tag{73}$$

Next, we consider the particles which diffuse collisionally in the parallel direction, but remain tied to the (perturbed) field lines. The particles will thus be dispersed in the x direction because the field lines do so. The particles' MSD (as a function of time) is estimated by replacing z by the parallel collisional MSD

$$z\to\hat{z}(t)=\langle\delta z^2(t)\rangle^{1/2}=(2\chi_\parallel t)^{1/2} . \tag{74}$$

One thus obtains the following result for the MSD, which can be expressed in terms of the SLT CTRW parameters:

$$\langle\delta x^2(t)\rangle_{\rm RR}=\left[\frac{\pi}{2}\right]^{1/2}\langle\delta x^2(t)\rangle=\sqrt{2}\sigma^2\left[\frac{t}{\tau_D}\right]^{1/2} . \tag{75}$$

This result differs by a numerical factor $\sqrt{\pi/2}$ from the result (22) obtained from the V-Langevin equation. The reason of this (slight) discrepancy is the following. The Rechester-Rosenbluth (RR) result is obtained by substituting an asymptotic result $[\hat{z}(t)]$ into an asymptotic result $[\langle \delta x^2(z) \rangle]$. Instead, the derivation of Eq. (22) involves a single asymptotic limit of the exact expression of $\langle \delta x^2(t) \rangle$. We now introduce the change of variables (74) in Eq. (72), which is thus transformed into an equation of evolution in time:

$$\partial_t n_{\rm RR}(x,t) = D_{\rm RR}(t) \nabla_x^2 n_{\rm RR}(x,t) \ , \qquad (76)$$

where $n_{\rm RR}(x,t) = \bar{n}[x, z(t)]$ and $D_{\rm RR}(t)$ is given by

$$D_{\rm RR}(t) = \frac{1}{2\sqrt{2}} \frac{\sigma^2}{\tau_D} \left[\frac{t}{\tau_D} \right]^{-1/2} = \left[\frac{\pi}{2} \right]^{1/2} D(t) \ . \qquad (77)$$

Thus the Rechester-Rosenbluth argument leads to a diffusion equation with a time-dependent diffusion coefficient $D_{\rm RR}(t)$.

In conclusion, the Rechester-Rosenbluth argument yields (up to the factor $\sqrt{\pi/2}$) the same MSD and the same running diffusion coefficient as the exact solution of the Langevin equation and as the SLT CTRW model. Nevertheless, the equation of evolution (76) is completely different from the equation (70) resulting from the SLT CTRW description. Equation (76) is a true Markovian diffusion equation with a time-dependent diffusion coefficient. By contrast, Eq. (70) is a non-Markovian diffusion equation. It may therefore be expected that the density profile predicted by Eq. (76) would be quite different from Eq. (70).

Equation (76) is indeed very easily solved (the change of variable t to T^2 transforms it into a diffusion equation with a constant coefficient). The solution, expressed in terms of the similarity variable q [Eq. (69)], is

$$n_{\rm RR}(x,t) = \frac{1}{\sigma} \frac{1}{(2\pi^2)^{1/4}} \left[\frac{\tau_D}{t} \right]^{1/4} \exp\left[-\frac{q^2}{2\sqrt{2}} \right] . \qquad (78)$$

Thus the profile obtained by the Rechester-Rosenbluth argument is quite different from the one predicted by the SLT CTRW Eq. (68). The former is simply a Gaussian packet whose width increases more slowly in time than the "usual" diffusive Gaussian (30) ($\sim \sqrt{t}$ instead of $\sim t$). Nevertheless, Eqs. (68) and (78) have an important feature in common: they both are of the general scaling form (37) for $d = 1$ and $\alpha = \frac{1}{2}$; they differ only by the form of the function $F(q)$. Thus, as stated before, the density profile is a much more sensitive description of the model than the MSD.

It is not difficult to exhibit the precise relationship between the SLT and the RR profiles or, equivalently between Eqs. (70) and (76) of which they are the fundamental solutions. Starting from the former, we may perform a very well-known *Markovianization procedure,* which amounts to neglecting the effect of the past history. For full generality, we start from Eq. (50) and approximate $n(\mathbf{x}, t-\tau)$ by $n(\mathbf{x},t)$ in the integral on the right-hand side. Using also Eq. (49), the non-Markovian equation is transformed into a true diffusion equation

$$\partial_t n(\mathbf{x},t) = D_0[1 - A(t)] \nabla^2 n(\mathbf{x},t) \ , \qquad (79)$$

where

$$1 - A(t) = 1 - \frac{1-\alpha}{\tau_D \Gamma(\alpha)} \int_{\tau_{\min}}^t d\tau \left[\frac{\tau_D}{\tau} \right]^{2-\alpha}$$
$$= 1 - \frac{1}{\Gamma(\alpha)} \left[\frac{\tau_D}{\tau_{\min}} \right]^{1-\alpha} + \frac{1}{\Gamma(\alpha)} \left[\frac{\tau_D}{t} \right]^{1-\alpha} .$$

The two first terms on the right-hand side cancel each other exactly for the value (52) of the cutoff $\tau_{\min}$ and we are left with

$$\partial_t n_M(\mathbf{x},t) = \frac{D_0}{\Gamma(\alpha)} \left[\frac{\tau_D}{t} \right]^{1-\alpha} \nabla^2 n_M(\mathbf{x},t) \ . \qquad (80)$$

We thus find in this Markovian approximation a diffusion equation with a time-dependent diffusion coefficient. It is easily checked that in the case $d = 1$ and $\alpha = \frac{1}{2}$, this equation reduces exactly to (76) and (77). We also write down the fundamental solution of Eq. (80), which is obtained by the same procedure as Eq. (78):

$$n_M(\mathbf{x},t) = \frac{1}{\sigma} \left[\frac{\Gamma(1+\alpha)}{2\pi} \right]^{1/2} \left[\frac{\tau_D}{t} \right]^{\alpha/2} \times \exp\left[-\frac{\Gamma(1+\alpha)}{2} q^2 \right] . \qquad (81)$$

This function has again the correct scaling form (37) and reduces to Eq. (78) for $d = 1$, $\alpha = \frac{1}{2}$. It is, however, quite different from the density profile (41) obtained from the SLT CTRW model.

We have thus identified precisely the nature of the approximation corresponding to the Rechester-Rosenbluth argument. It should now be noted that the Markovian approximation is usually performed as an asymptotic approximation for integral equations in which the kernel is a rapidly decaying function, typically of exponential type. The latter defines a characteristic memory time τ_m, whereas the characteristic time of the variation of the density profile is, say τ_r. Whenever $\tau_m \ll \tau_r$, the Markovian approximation is fully justified. But in the present case, the decay of the memory kernel is very slow; it is characterized by the power-law long tail $\sim t^{2-\alpha}$. There exists actually no definite characteristic time τ_m, hence the Markovianization can hardly be justified. It may indeed also be noted that the "Markovianized" profile (81) cannot be obtained from the exact profile (41) by a limiting process.

The density profiles (68) [produced by Eq. (70)] and (78) [produced by Eq. (76)] are compared in Fig. 1. They are drawn as functions of x at various times. We see that the SLT profiles are much lower in the bulk, but extend much farther along x (long tail). The width of both curves grows in time, proportionally to $t^{1/4}$ in both cases. The shapes of the profiles are clearly quite different, even though the MSD has the same scaling.

We now briefly discuss a density profile that, at first sight, appears to be quite different from both previous ones. Rax and White [7] considered the V-Langevin equations (2) and (3) for the problem of magnetic turbulence and gave a solution not only for the MSD but also for the density profile. The latter was obtained by

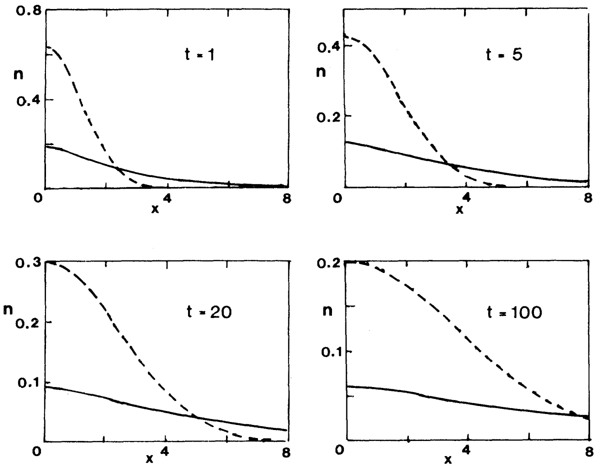

FIG. 1. Density profiles at various times in reduced units of t/τ_D and x/σ. Full line, SLT CTRW, Eq. (68); dashed line: Markovian approximation, Eq. (78).

assigning a Gaussian probability distribution to the various paths and using a functional integration technique together with an asymptotic saddle point approximation. It must be stressed that they solved the *two-dimensional problem* $d=2$ and calculated the profile as a function of t and $r_\perp=(x^2+y^2)^{1/2}$. Omitting all constant factors, we write their result as

$$n_{\rm RW}(r_\perp,t)\sim t^{-1/3}r_\perp^{-2/3}\exp\left[-\frac{r_\perp^{4/3}}{t^{1/3}}\right] . \qquad (82)$$

This is a quite different profile from both previous ones. It has a peculiar singularity in $r_\perp=0$. It should, however, be noted that the domain of small $r_\perp$ is not relevant because (82) is an asymptotic formula. The authors did not notice that Eq. (82) is just a particular case of the general scaling relation (40). Indeed, when we change the variable $r_\perp$ into the scaling variable q, defined by Eq. (38) with $\alpha=\frac{1}{2}$, we find

$$n_{\rm RW}(q,t)\sim t^{-1/2}q^{-2/3}\exp(-q^{4/3}), \quad d=2 . \qquad (83)$$

This is precisely of the form of (37), i.e., $t^{-\alpha d/2}F(q)$. Thus the Rax-White result shows that the V-Langevin equations (in two dimensions), combined with their probabilistic assignment, lead to the same density profile as the SLT random walk. This important result will be discussed further in Sec. VII.

It is easily checked that the solution of the Markovianized equation (80) in two dimensions is very similar to Eq. (78), except for the explicit time dependence, which is $\sim t^{-1/2}$ instead of $t^{-1/4}$. More precisely, this is the solution depending on $r\equiv(x^2+y^2)^{1/2}$ in radial coordinates:

$$n_{\rm RR2}(r,t)=\frac{1}{4\pi\sqrt{t}}\exp(-\tfrac{1}{4}q^2), \quad d=2 . \qquad (84)$$

VII. KINETIC EQUATION vs SLT CTRW

In the preceding section we investigated the possibility of adapting a SLT CTRW model to the magnetic turbulent diffusion problem, by choosing appropriate parameters α $(=\frac{1}{2})$, σ, and τ_D in order to match the MSD calculated from the V-Langevin equations. We saw that this choice does not uniquely determine the density profile. Only in the two-dimensional Rax-White case is there an attempt to relate the profile to the V-Langevin equations; but the methods used by these authors introduces additional assumptions, which have to be evaluated.

In order to treat this problem in full generality we solve a kinetic equation for the one-particle distribution function. Though being equivalent to the V-Langevin equations, its solution yields directly the complete density profile rather than its individual moments. In Ref. [9] we introduced the concept of a *hybrid kinetic equation* (HKE) constructed in such a way as to contain the same physics as the Langevin equations. This is a first-order partial differential equation for the distribution function $f(x,z;t)$, whose characteristics are the V-Langevin equations (2) and (3) for a given realization of both the magnetic field $b(z)$ and the velocity $v(t)$:

$$\frac{\partial}{\partial t}f+v(t)\frac{\partial}{\partial z}f+v(t)b(z)\frac{\partial}{\partial x}f=0 . \qquad (85)$$

This is a simpler version of the HKE than the one studied in Ref. [9], which was adapted to the Langevin equations for the acceleration. It was shown there that the latter equations (as well as the HKE) yield the same MSD as the V-Langevin equations.

The HKE looks rather different from the usual kinetic equations of statistical physics, which describe the distribution function in the complete phase space $(\mathbf{x},\mathbf{v})$. The function f appearing here depends only on two spatial coordinates and not on the velocity. The parallel velocity $v(t)$ is considered a given function of time. The HKE is declared to be a doubly stochastic equation: both the velocity $v(t)$ and the magnetic field $b(z)$ are considered Gaussian random variables defined statistically by Eqs. (4)–(10). The strategy of solution is straightforward. The HKE is integrated exactly for a given realization. This solution is used for the calculation of the relevant quantities (such as the particle flux, the density profile, etc.). The final result is then averaged over v and b. This operation does not require any additional probabilistic assumptions beyond those underlying the V-Langevin equations.

We note that the average of the distribution function $f(x,z;t)$ is nothing other than the density profile

$$n(x,t)=\langle\langle f(x,z;t)\rangle\rangle_{v,b} . \qquad (86)$$

It will be shown below that, with the assumptions (4)–(10), this quantity is independent of z. The distribution function is decomposed as

$$f(x,z;t)=n(x,t)+\delta f(x,z;t) . \qquad (87)$$

The two terms on the right-hand side vary in time on different scales, the density profile having a much slower dependence. This function obeys the average kinetic equation

$$\partial_t n(x,t) + \frac{\partial}{\partial x}\langle\langle v(t)b(z)\delta f(x,z;t)\rangle\rangle_{v,b} = 0 \ . \quad (88)$$

(It will be seen that, because of the spatial homogeneity in z, the average in the second term is independent of z.) This equation for the average density profile $n(x,t)$ is coupled to the fluctuating distribution $\delta f(x,z;t)$. The aim of our treatment (as in any kinetic theory) will be to eliminate the latter and to obtain a closed equation for the average density profile. The fluctuating distribution obeys the following equation, derived from (85):

$$\partial_t \delta f + v(t)\frac{\partial}{\partial z}\delta f + v(t)b(z)\frac{\partial}{\partial x}\delta f = -v(t)b(z)\frac{\partial}{\partial x}n(x,t) \ . \quad (89)$$

Anticipating a check *a posteriori*, we have eliminated from Eq. (89) all the terms that do not contribute to the average in Eq. (88).

Equation (89) is a linear, first-order, inhomogeneous partial differential equation for δf for any given realization. It is, however, *stochastically nonlinear*. Its solution or, equivalently, the propagator, is easily obtained by the method of characteristics:

$$G(x,z,t|x',z',t') = \delta[x - F_2(t,t')]\delta[z - F_1(t,t')] \ , \quad (90)$$

where

$$F_1(t,t') = \int_{t'}^{t} d\theta\, v(\theta) \ ,$$
$$F_2(t,t') = \int_{t'}^{t} d\theta\, v(\theta) b[z - F_1(t,\theta)] \ . \quad (91)$$

Following the classical procedure of Ref. [9], we obtain the solution of Eq. (89) (neglecting the solution of the homogeneous equation) and substitute it into Eq. (88):

$$\partial_t n(x,t) = \frac{\partial}{\partial x}\int_0^t d\tau \left\langle\!\!\left\langle v(t)b(z)v(\tau)b[z - F_1(t,\tau)]\frac{\partial}{\partial x}n(x - F_2(t,\tau),\tau)\right\rangle\!\!\right\rangle_{v,b} \ . \quad (92)$$

This equation can be transformed by using the Fourier representation (6) and the explicit forms (91):

$$\partial_t n(x,t) = (2\pi)^{-2}\int_0^t d\tau \int dk_1 dk_2 \left\langle\!\!\left\langle b(k_1)b(k_2)v(t)v(\tau)\exp\left[-i(k_1+k_2)z + ik_2\int_\tau^t d\theta\, v(\theta)\right]\right.\right.$$
$$\left.\left. \times \nabla_x^2 n\left\{x - \int_\tau^t d\xi\, v(\xi)(2\pi)^{-1}\int d\kappa\, b(\kappa)\exp\left[i\kappa\left[z - \int_\xi^t d\lambda\, v(\lambda)\right]\right],\tau\right\}\right\rangle\!\!\right\rangle_{v,b} \ . \quad (93)$$

This is the *exact equation of evolution* for the density profile resulting from the hybrid kinetic equation (88). As can be seen, this is a quite involved integro-differential equation for the density profile; it is, however, a *closed* equation. The main complication comes from the displacement of the variable x by the quantity $F_2(t,\tau)$. The latter depends on both stochastic functions $v(t)$ and $b(z)$. As a result, the density profile cannot be taken out of the average on the right-hand side of Eq. (93). It is also clear that, in spite of some similarities, Eq. (92) is *not* of the form of an equation resulting from a SLT CTRW process. Nevertheless, upon multiplication by x^2 and integration over x, this equation yields the correct running diffusion coefficient (12), as can easily be checked. This shows that the determination of the density profile is a much more complex problem than the determination of the MSD. Indeed, the MSD is but one among the infinite number of moments necessary for the definition of the profile.

We now introduce some simplifying approximations. A very commonly used approximation in plasma turbulence theory is the *quasilinear approximation*. It consists of retaining on the right-hand side of Eq. (93) only contributions of first order in the spectrum $\mathcal{B}(k)$ defined in Eq. (7). It is easily seen that this approximation amounts to neglecting altogether the displacement $F_2(t,\tau)$ in x in Eq. (93), i.e., getting rid of the main difficulty. In the remaining expression, the average over b factorizes out from the average over v. Using Eq. (7), we introduce a factor $\delta(k_1+k_2)$, which allows the integration over k_1 to be done: this shows (as expected) that the average in the equation is indeed independent of z. The result of these operations is

$$\partial_t n(x,t) = \int_0^t d\tau\, H(\tau)\nabla_x^2 n(x,t-\tau) \ , \quad (94)$$

with

$$H(\tau) = (2\pi)^{-2}\int dk\, \mathcal{B}(k)\mathcal{Z}(k,\tau) \ . \quad (95)$$

The function $\mathcal{Z}(k,\tau)$ is precisely the function defined in Eq. (13). Equation (94) has exactly the same form as the non-Markovian diffusion equation (45) resulting from a SLT CTRW process.

Equation (94) is the main result of this work. The hybrid kinetic equation describing magnetic turbulence in the quasilinear approximation is equivalent to a SLT CTRW process. This result also shows indirectly that the approximations used in the derivation of the Rax-White result (82) are equivalent to the quasilinear approximation.

Equation (94) will be further discussed below. Meanwhile, we introduce a further approximation, namely, the *Markovian approximation*. As explained in Sec.

VI, this amounts to replacing $n(x,t-\tau)$ by $n(x,t)$ in Eq. (94), which reduces to

$$\partial_t n(x,t)=D(t)\nabla_x^2 n(x,t) . \quad (96)$$

This is an ordinary diffusion equation with a time-dependent diffusion coefficient, of the same form as Eq. (76), with the diffusion coefficient given by

$$D(t)=(2\pi)^{-2}\int_0^t d\tau \int dk\, \mathcal{B}(k) Z(k,\tau) . \quad (97)$$

This is exactly the coefficient (12) and (19) obtained from the V-Langevin equations and also from the Markovianized SLT CTRW in Eq. (77). (Note that the factor $\sqrt{\pi/2}$ in the latter equation has disappeared here because we took the Markovian limit of the exact evolution equation.)

We now return to the non-Markovian diffusion equation (94). The memory kernel $H(\tau)$ can be calculated analytically by evaluating the integral over k: this was done in Sec. II. The result is simply $H(\tau)=\epsilon F(\nu\tau)$, where the function $F(x)$ is defined in Eq. (15). A more interesting form is obtained by using dimensionless variables, reduced with the SLT CTRW parameters defined in Eq. (67):

$$\xi=\frac{x}{\sigma}, \quad \theta=\frac{t}{\tau_D}=\gamma\nu t . \quad (98)$$

The non-Markovian diffusion equation (94) is then written in the form

$$\partial_\theta n(\xi,\theta)=\int_0^\theta d\bar{\theta}\, G(\bar{\theta};\gamma)\nabla_\xi^2 n(\xi,\theta-\bar{\theta}) , \quad (99)$$

with

$$G(\theta;\gamma)=\frac{1}{2\sqrt{\pi}\gamma}\frac{1}{[1+\gamma\psi(\theta/\gamma)]^{1/2}} \times\left[e^{-\theta/\gamma}-\frac{\gamma}{2}\frac{\varphi^2(\theta/\gamma)}{1+\gamma\psi(\theta/\gamma)}\right] . \quad (100)$$

This memory kernel possesses many interesting features. First we note that $G(\theta,\gamma)$ is a regular function, devoid of any singularities over the whole range $0\le\theta<\infty$. For small values of θ, i.e., $\theta\ll\gamma$, the function can be expanded and yields

$$2\sqrt{\pi}G(\theta,\gamma)=\frac{1}{\gamma}-\frac{1}{\gamma^2}\theta+\left[\frac{1}{2\gamma^3}-\frac{1}{\gamma^2}\right]\theta^2+\cdots , \quad \theta\ll\gamma . \quad (101)$$

This is in contrast to the kernel $\phi(t)$ appearing in the SLT CTRW non-Markovian diffusion equation (45) and (47), which becomes infinite as $t\to 0$. The reason is in the fact that Eq. (99) originates from a semidynamical model, which covers the evolution over the whole range of times. On the contrary, Eq. (47) was derived as an *asymptotic approximation*, valid only for times that are long compared to the characteristic time τ_D of the CTRW. Moreover, as the asymptotic decay is very slow (power law), the equation will be accurate only for "very long" times.

We also consider the large-θ approximation:

$$G(\theta,\gamma)\sim-\frac{1}{4\sqrt{\pi}\theta^{3/2}}, \quad \theta\gg\gamma . \quad (102)$$

The function $G(\theta,\gamma)$ is shown in Fig. 2 for three different values of the parameter γ. (This function is actually the same as the function $F(x)$ [Eq. (15)], except that the scaling of the variable θ is different.) The asymptotic form (102) is compared to the exact function in Fig. 3.

Another conspicuous feature of the memory kernel is its dependence on γ. Consider, indeed, the SLT non-Markovian diffusion equation (70). If we introduce the nondimensional variables $\theta=t/\tau_D$ and $\xi=x/\sigma$ the equation reduces to

$$\partial_\theta n_{\rm SLT}(\xi,\theta)=\tfrac{1}{2}\nabla_\xi^2 n(\xi,\theta) -\frac{1}{4\sqrt{\pi}}\int_{\pi^{-1}}^\theta d\bar{\theta}\,\bar{\theta}^{-3/2}\nabla_\xi^2 n_{\rm SLT}(\xi,\theta) . \quad (103)$$

This equation contains no parameter at all. In other words, the SLT CTRW process depends only on two dimensional constants τ_D and σ, which are used to scale the variables t and x, respectively. In contrast, the solution of the HKE contains an additional parameter γ, which measures the collisionality. Thus this feature together, with the different shape of the memory function, in particular the convergence at $\theta=0$, shows that the SLT CTRW model cannot be equivalent to the quasilinear HKE for all times. However, it is clear that a com-

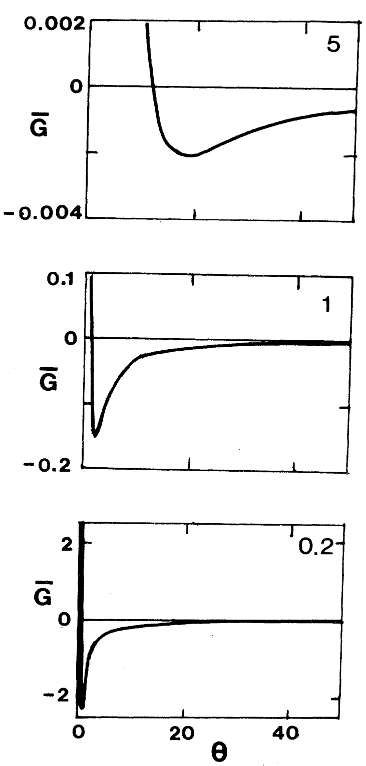

FIG. 2. Memory function $\bar{G}=2\sqrt{\pi}G(\theta,\gamma)$ [Eq. (100)] for $\gamma=5,1,0.2$. (The vertical scale is expanded for visibility.)

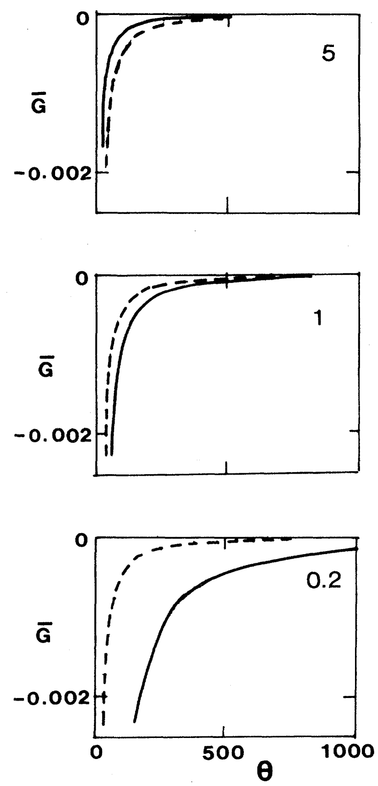

FIG. 3. Large-θ approximation to the memory function $\bar{G}(\theta,\gamma)$: Full line, exact, Eq. (100); dashed line Eq. (102).

parison of the two models *over the whole range of times* is unfair: indeed, the SLT CTRW model is explicitly an asymptotic approximation, valid for $\theta \gg 1$, and the comparison should only be made in this range. It appears from Eq. (102) that for $\theta \gg 1$, the memory function $G(\theta,\gamma)$ becomes independent of γ and reduces to $\theta^{-3/2}$, i.e., it becomes identical (up to a numerical factor) to the SLT CTRW memory function.

We now try to construct a convenient asymptotic approximation to Eq. (99). We pick a cutoff time $\theta_{\min}$ whose value will be determined *a posteriori* and divide the integration domain in Eq. (99) in two parts. Assuming $\theta \gg \theta_{\min}$, we may approximate $n(\xi,\theta-\bar{\theta}) \to n(\xi,\theta)$ in the integral taken from 0 to $\theta_{\min}$. On the other hand, we assume that in the integral from $\theta_{\min}$ to ∞, we may use the asymptotic (γ-independent) approximation $G(\bar{\theta},\gamma) \to -(4\sqrt{\pi}\theta^{3/2})^{-1}$. Thus

$$\partial_\theta n(\xi,\theta) = \int_0^{\theta_{\min}} d\bar{\theta}\, G(\bar{\theta},\gamma) \nabla_\xi^2 n(\xi,\theta) - \frac{1}{4\sqrt{\pi}} \int_{\theta_{\min}}^{\theta} d\bar{\theta}\, \bar{\theta}^{-3/2} \nabla_\xi^2 n(\xi,\theta-\bar{\theta}) \ . \quad (104)$$

In the first term the integration involves only the memory function. As the latter converges at $\theta=0$, the integral is merely a constant depending on the parameters γ and $\theta_{\min}$: it can be calculated analytically [9]. The result, in reduced variables, is

$$K(\theta_{\min},\gamma) \equiv \int_0^{\theta_{\min}} d\bar{\theta}\, G(\bar{\theta},\gamma) = \frac{1}{2\sqrt{\pi}} \frac{\varphi(\theta_{\min}/\gamma)}{[1+\gamma\psi(\theta_{\min}/\gamma)]^{1/2}} \ . \quad (105)$$

Equation (104) is now written as

$$\partial_\theta n(\xi,\theta) = K(\theta_{\min},\gamma) \nabla_\xi^2 n(\xi,\theta) - \frac{1}{4\sqrt{\pi}} \int_{\theta_{\min}}^{\theta} d\bar{\theta}\, \bar{\theta}^{-3/2} \nabla_\xi^2 n(\xi,\theta-\bar{\theta}) \ . \quad (106)$$

This equation is very similar to the SLT equation (103). We must now determine the cutoff time $\theta_{\min}$: the procedure is similar to the one used in Sec. V. We multiply both sides of Eq. (106) by ξ^2 and integrate over ξ

$$\partial_\theta \langle \xi^2(\theta) \rangle = 2K(\theta_{\min},\gamma) - \frac{1}{\sqrt{\pi}\theta_{\min}^{1/2}} + \frac{1}{\sqrt{\pi}} \theta^{-1/2} \ . \quad (107)$$

The last term equals precisely the properly reduced asymptotic value (22). We thus determine the cutoff time by annulling the first two terms

$$2\sqrt{\pi} K(\theta_{\min},\gamma) = \theta_{\min}^{-1/2} \ . \quad (108)$$

Substituting Eq. (105) into Eq. (108), squaring the latter, and introducing the notation $m = \theta_{\min}/\gamma$, we find the following equation for the determination of m:

$$\Phi(m,\gamma) \equiv -(1+2m)e^{-m} + me^{-2m} + 1 - \gamma^{-1} = 0 \ . \quad (109)$$

This transcendental equation must be solved for m: this has to be done numerically. We note first that

$$\Phi(0,\gamma) = -\frac{1}{\gamma} < 0 \ .$$

On the other hand, the function tends asymptotically towards

$$\lim_{m\to\infty} \Phi(m,\gamma) = 1 - \frac{1}{\gamma} \ .$$

The numerical solution shows that, after a shallow minimum, the function $\Phi(m,\gamma)$ increases monotonically to its asymptotic value. Therefore, if $\gamma > 1$, the asymptote is positive and the curve $\Phi(m,\gamma)$ crosses the m axis in order to join the negative initial value to the positive final value: there is a real root to the equation. If, however, $\gamma < 1$, the curve remains below the m axis and there is no root. For $\gamma = 1$, the curve is asymptotically tangent to the m axis (see Fig. 4). The numerical solution of Eq. (109) is plotted in Fig. 5. As expected, there is no solution for $\gamma < 1$, i.e., for highly collisional plasmas. The solution is infinite for $\gamma = 1$. For $\gamma > 2$, the solution is surprisingly close to linear. A very good fit is

$$\theta_{\min} = 2.848 + 0.976\gamma, \quad \gamma > 2 \ . \quad (110)$$

In conclusion, for weakly collisional plasmas $\gamma > 1$, there exists a real cutoff $\theta_{\min}$ and hence a finite value for the number $K(\theta_{\min},\gamma)$. Equation (106), which has the same form as the SLT CTRW equation (103) is thus validated as an asymptotic approximation.

We now note that for *strongly collisional plasmas*

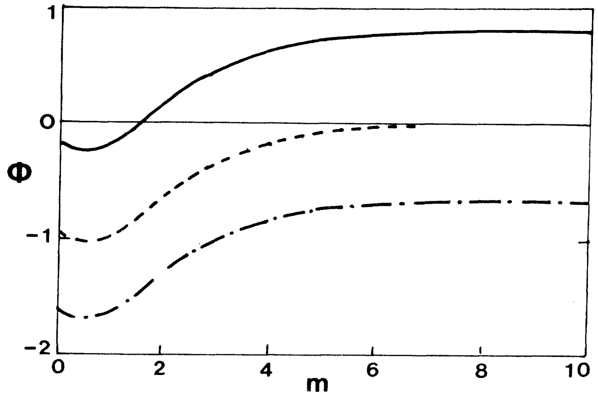

FIG. 4. Function $\Phi(m,\gamma)$ for three values of γ.

$\gamma \ll 1$, a similar (though less precise) asymptotic approximation can be constructed. In this case, as can be seen from Fig. 2, the memory function is very strongly peaked at the origin. From Eq. (101) we find $G(0,\gamma) = (2\sqrt{\pi}\gamma)^{-1}$, which tends to infinity when $\gamma \to 0$. We may thus use a rough approximation of the "δ plus tail" type:

$$G(\theta,\gamma) \approx \frac{1}{2}\delta(\theta) - \frac{1}{4\sqrt{\pi}}\theta^{-3/2}\Theta(\theta-\theta_{\min}) , \qquad (111)$$

where $\Theta(x)$ is the Heaviside step function. When this form is substituted into Eq. (99), the result is exactly the SLT CTRW equation (103). In conclusion, for both weakly collisional and very strongly collisional plasmas, the SLT CTRW model provides a very good asymptotic representation of the exact quasilinear equation for the density profile.

VIII. CONCLUSIONS

The CTRW concept appears as a quite useful tool for the investigation of complex, in particular partially disordered, systems. As such, it appears quite promising for the modeling of transport in turbulent plasmas. The problem studied here is deliberately oversimplified, but has the advantage of being exactly soluble. It therefore yields the possibility of a direct comparison with the CTRW model.

The latter model was further developed here for the case of the SLT CTRW. The non-Markovian diffusion equation (50) provides a very clear physical picture of the evolution. A balance between the purely diffusive term and the non-Markovian one limited by an inferior cutoff time allowed us to construct a convergent equation that yields correct values for all the moments and hence for the complete density profile. As the memory function extends very far into the past (long tail), a Markovian approximation is very questionable.

As shown in Sec. VI, it is very easy to construct a SLT CTRW model that yields the exact MSD for the magnetic turbulence problem. It is much less trivial to evaluate the validity of the model for the description of the complete density profile. It was shown that the semiquantitative argument of Rechester and Rosenbluth [5] yields only the Markovianized limit of the equation of evolution (76). The corresponding solution (78) has a Gaussian shape that cannot be recovered as a limit of the exact solution (68). On the other hand, the solution of the V-Langevin equation in two dimensions, obtained by Rax and White [7], has exactly the scaling predicted by the SLT CTRW model. Their solution involves a number of assumptions, whose meaning appears more clearly from the results of Sec. VII.

It was shown there that the hybrid kinetic equation (equivalent to the V-Langevin equations) can be solved exactly in the present case. The exact density profile is definitely not describable by a simple CTRW model because of its strong nonlinear dependence on the fluctuation spectrum. The quasilinear approximation (widely used for weak plasma turbulence) yields, however, a non-Markovian diffusion equation (100), equivalent to a CTRW. Its memory kernel is convergent over the whole range of times and depends asymptotically on time as $t^{-3/2}$, like the SLT CTRW. It contains, however, an additional dimensionless parameter γ, describing the collisionality of the plasma. It appears that an accurate modelization by a specific SLT CTRW of the form of (50) (in particular the determination of a cutoff) is only possible in a regime of weak collisionality ($\gamma \gg 1$). A somewhat rougher approximation of the same type is, however, possible also in the domain of very strong collisionality.

In conclusion, our results show that the continuous time random walk concept is a useful and interesting tool in anomalous transport theory for turbulent plasmas. We intend to apply it to the study of more realistic and more complex problems.

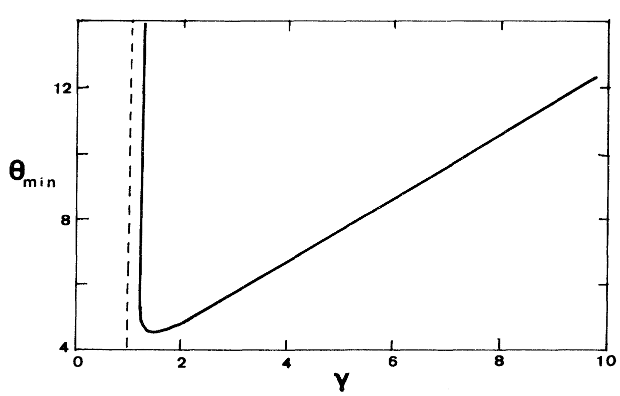

FIG. 5. Cutoff time $\theta_{\min}$ vs γ.

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